Def^n

Let X be a non-empty set. A function

$$d : X \times X \to \mathbb{R}$$

is said to be a metric on X if it satisfies the following conditions:

i) $d(x, y) \geq 0, \ \forall \ x, y \in X$ (Positiveness)

ii) $d(x, y) = 0 \Leftrightarrow x = y, \ x, y \in X$

iii) $d(x, y) = d(y, x), \ \forall \ x, y \in X$ (Symmetry)

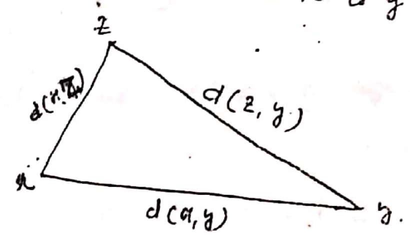
iv) $d(x, y) \leq d(x, z) + d(z, y), \ \forall \ x, y, z \in X$ (triangular inequality)

The orderd pair (X, d) is called a metric space. If there is no confusion likely to occur we, sometimes, denoted the metric space (X, d) by X.

Note:⇒ A metric d is also called a distance function and the non-negative real number $d(x, y)$ is to be thought of as the distance between x and y.

Remarks:-

1. The triangle inequality my be interpreted as that "the length of one side of a triangle can not exceed the sum of the length of the other two sides". Equivalently, the distance from x to y via any intermediate point z can not be shorter than the direct distance from x to y



2. The triangle inequality can be generalised for any number of additional points $z_1, z_2, \ldots z_n$ in X; ie,

$$d(x, y) \leq d(x, z_1) + d(z_1, z_2) + \cdots + d(z_n, y)$$

Semi-Metric space:-

A function $d : X \times X \to \mathbb{R}$ which satisfies (i), (iii), (iv) is called a semi-metric and (X, d) is called a semi-metric space.

Ex.1 Let $X = \mathbb{R}$, the set of all real numbers. For $x, y \in X$, define $d(x, y) = |x-y|$

Then (X, d) is a metric space. This is called the metric space $\mathbb{R}$ with the usual metric and denoted it by $\mathbb{R}_u$.

Soln: i) We have $d(x, y) = |x-y|$, $x, y \in X$. —— ①

Here it is clear that $d(x, y) \geq 0$

as $d(x, y) = |x-y| \geq 0 \quad \forall\, x, y \in X$.

ii) When $y = x$, we have from ①

$d(x, x) = |x-x| = 0$

When $d(x, y) = 0$

$\therefore |x-y| = 0$

$\Rightarrow x = y$

$\therefore d(x, y) = 0 \iff x = y, \quad \forall\, x, y \in X$.

iii) $d(x, y) = |x-y|$

$= |-(y-x)|$

$= |y-x| = d(y, x)$

$\therefore d(x, y) = d(y, x), \quad \forall\, x, y \in X$.

iv) For any $x, y, z \in X$, we have by triangle inequality

$|x-y| \leq |x-z| + |z-y|$

or, $d(x, y) \leq d(x, z) + d(z, y)$.

$\therefore d(x, y)$ satisfies all the properties of distance function. So $d(x, y)$ is a distance function or metric on X.

Hence (X, d) is a metric space.

1. Q Let $X = \mathbb{C}$, the set of all complex numbers. For $x, y \in X$, define $d(x,y) = |x-y|$. then (X, d) is a metric space. this is called a metric space on $\mathbb{C}$ with the usual metric and we denoted it by $\mathbb{C}_u$. 2H-12

Soln: we have $d(x,y) = |x-y|$, $x, y \in \mathbb{C}$ ——— ①

i) $d(x,y) = |x-y| \geq 0 \quad \forall x, y \in \mathbb{C}$.

ii) Also $d(x,y) = 0$ iff $x = y$

iii) $d(x,y) = |x-y| = |-(y-x)|$
$= |y-x| = d(y,x). \quad \forall x, y \in \mathbb{C}$

iv) we have for any $z_1, z_2 \in \mathbb{C}$,

$|z_1 + z_2| \leq |z_1| + |z_2|$ ——— ②

putting $z_1 = x - z$
$z_2 = z - y$ in (2).

$\therefore |x - z + z - y| \leq |x-z| + |z-y|$

or, $|x-y| \leq |x-z| + |z-y|$

or, $d(x,y) \leq d(x,z) + d(z,y)$

$\therefore d(x,y)$ Satisfies all the properties of distance function. So $d(x,y)$ is a distance function or metric.

Hence (X, d) is a metric space.

Ex: Let W denote the space of all sequences in $\mathbb{K}$. define $d(x,y) = \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n(1+|x_n-y_n|)}$.

where $x = \{x_n\}$ and $y = \{y_n\}$ are in W. prove ~~that~~ d defines a metric on W.

UH-10

Ex: 8 Let X be an arbitrary non-empty set. For x, y define d by

$$d(x,y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$$

Then (X,d) is a metric space. The metric d is called the discrete metric and the space (X,d) is called discrete metric space and is denoted by X_d.

Soln:- Here $d(x,y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$

i) $d(x,y) \geq 0 \;\; \forall \; x, y \in X$

ii) $d(x,y) = 0$, when $x = y$.

Let $d(x,y) = 0$, then by definition we have $x = y$.

$\therefore \; d(x,y) = 0$ iff $x = y$

iii) $d(y,x) = \begin{cases} 0, & y = x \\ 1, & y \neq x. \end{cases}$

$d(x,y) = \begin{cases} 0, & x = y \\ 1, & x \neq y. \end{cases}$

$\therefore \; d(x,y) = d(y,x)$.

iv)

	$d(x,y)$	$d(x,z)$	$d(y,z)$	$d(x,y) + d(y,z)$
$x = y = z$	0	0	0	0
$x \neq y = z$	1	1	0	1
$z \neq x = y$	0	1	1	1
$y \neq z = x$	1	0	1	2
$x \neq y \neq z$	1	1	1	2

From this table we can see that

$d(x,z) \leq d(x,y) + d(y,z)$, in all cases.

$\therefore \; d(x,z) \leq d(x,y) + d(y,z)$ holds $\forall \; x, y, z \in X$

$\therefore \; (X,d)$ is a metric space.

Let $X = \mathbb{R}^2$, [illegible]

i.e. $d^\infty(x,y) = \max\{|x_1-y_1|, |x_2-y_2|\} = \max_{1\le i\le 2}\{|x_i-y_i|\}$

& $d^1(x,y) = |x_1-y_1| + |x_2-y_2| = \sum_{i=1}^{2}|x_i-y_i|$

[illegible] $(\mathbb{R}^2, d^\infty)$ and $(\mathbb{R}^2, d^1)$ [illegible] metric space.

Sol:-

(i) We have $d^\infty(x,y) = \max\{|x_1-y_1|, |x_2-y_2|\} \geq 0$

$\forall\ x = (x_1, x_2)$
$y = (y_1, y_2) \in \mathbb{R}^2$

(ii) $d(x,x) = \max\{|x_1-x_1|, |x_2-x_2|\}$
$= 0$

Let $d(x,y) = 0$

i.e. $\max\{|x_1-y_1|, |x_2-y_2|\} = 0$

i.e. $|x_1-y_1| = 0$, and $|x_2-y_2| = 0$

i.e. $x_1 = y_1$ and $x_2 = y_2$

$\therefore d(x,y) = 0 \Leftrightarrow x = y$

(iii) $d(x,y) = \max\{|x_1-y_1|, |x_2-y_2|\}$

$= \max\{|-(y_1-x_1)|, |-(y_2-x_2)|\}$

$= \max\{|y_1-x_1|, |y_2-x_2|\}$

$= d(y,x)$

(iv) Let $x = (x_1, x_2)$, $y = (y_1, y_2)$, $z = (z_1, z_2) \in \mathbb{R}^2$

then $|x_1-z_1| \leq |x_1-y_1| + |y_1-z_1|$

$\leq \max(|x_1-y_1|, |x_2-y_2|) + \max(|y_1-z_1|, |y_2-z_2|)$

$= d^\infty(x,y) + d^\infty(y,z)$ —— ①

Similarly, $|x_2-z_2| \leq d^\infty(x,y) + d^\infty(y,z)$ —— ②.

Thus

$d^{*}(x,z) = \max(|x_1 - z_1|, |x_2 - z_2|)$

$\leq d^{*}(x,y) + d^{*}(y,z)$

$\therefore$ d^{*} satisfy all the conditions of the distance function. Hence (X, d^{*}) is a metric space.

Soln: (ii)

(i) We have $d^{**}(x,y) = |x_1 - y_1| + |x_2 - y_2| \geq 0$

$\forall\ x, y \in \mathbb{R}^2$.

(ii) $d(x,x) = |x_1 - x_1| + |x_2 - x_2| = 0$

Let $d(x,y) = 0$

i.e. $|x_1 - y_1| + |x_2 - y_2| = 0$.

i.e., $|x_1 - y_1| = 0$ and $|x_2 - y_2| = 0$

i.e., $x_1 = y_1$ and $x_2 = y_2$

$\Rightarrow x = y$.

$\therefore$ $d(x,y) = 0$ iff $x = y$

(iii) $d(x,y) = |x_1 - y_1| + |x_2 - y_2|$

$= |-(y_1 - x_1)| + |-(y_2 - x_2)|$

$= |y_1 - x_1| + |y_2 - x_2|$

$= d(y,x)$

$\therefore$ $d(x,y) = d(y,x)$

(iv) Let z be any point which represent as $z = (z_1, z_2) \in \mathbb{R}^2$.

$\therefore$ $d^{**}(x,z) = |x_1 - z_1| + |x_2 - z_2|$

$\leq |x_1 - y_1| + |y_1 - z_1| + |x_2 - y_2| + |y_2 - z_2|$

$= (|x_1 - y_1| + |x_2 - y_2|) + (|y_1 - z_1| + |y_2 - z_2|)$

$= d^{**}(x,y) + d^{**}(y,z)$

$\therefore$ $d^{**}(x,z) \leq d^{**}(x,y) + d^{**}(y,z)$

$\therefore$ $d^{**}(x,y)$ satisfy all the condn of the metric then the ordered pair (X, d^{**}) is a metric space

Scanned by CamScanner

Similarly we can show that the following functions are form a metric space.

(i) $d(x,y) = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2} = \left(\sum_1^2 (x_i-y_i)^2\right)^{1/2}$

(ii) $d(x,y) = \sqrt[3]{(|x_1-y_1|)^3+(|x_2-y_2|)^3}$

(iii) $d(x,y) = \sqrt[p]{|x_1-y_1|^p+|x_2-y_2|^p}$.

(iv) $d(x,y) = \sqrt[p]{(|x_1-y_1|)^p+|x_2-y_2|^p+|x_3-y_3|^p}$.

Here X is the set of all 3-tuple's i.e., all the points of the form (x_1,x_2,x_3), (y_1,y_2,y_3), (z_1,z_2,z_3) etc.

Th: Prove that a metric d is always non-negative.

Proof: For $x,y \in X$, it follows that

$d(x,y)+d(y,x) \geq d(x,x)$

i.e., $2d(x,y) \geq 0$, [Since $d(x,x)=0$, and $d(x,y)=d(y,x)$].

i.e., $d(x,y) \geq 0$.

Ex: 6 Let $X = \mathbb{Q}$, the set of all rational numbers. For $x,y \in X$, define $d(x,y) = |x-y|$

then prove that (X,d) is a metric space.

Ex: 7 Let $X = [0,1)$. For $x,y \in X$, define

$d(x,y) = |x-y|$

then prove that (X,d) is a metric space.

Ex: The set l_∞ of all bounded sequences $\{x_n\}$ of real numbers with the function d defined by $d(\{x_n\},\{y_n\}) = \sup\{|x_n-y_n| : n \in \mathbb{N}\}$, $\forall \{x_n\},\{y_n\} \in l_\infty$ is a metric on l_∞.

Hints:- For the triangle inequality, $\forall \{x_n\},\{y_n\},\{z_n\} \in l_\infty$

$|x_n-y_n| = |(x_n-z_n)+(z_n-y_n)| \leq |x_n-z_n|+|z_n-y_n|$

$\therefore \sup_n |x_n-y_n| \leq \sup_n |x_n-z_n| + \sup_n |z_n-y_n|$

Ex: 8 If d is a metric on a set X. then show that d_1 and d_2 are metric on X. where

VH-13

i) $d_1(x,y) = \frac{d(x,y)}{1+d(x,y)}$

ii) $d_2(x,y) = \min\{1, d(x,y)\}$.

Solⁿ:- (i) Since d is a metric then ~~we have~~ it satisfy the following conditions

i) $d(x,y) \geq 0 \quad \forall x,y \in X$

ii) $d(x,y) = 0$ iff $x = y$

iii) $d(x,y) = d(y,x)$, $\forall x,y \in X$

iv) $d(x,y) \leq d(x,z) + d(z,y)$, $\forall x,y,z \in X$.

proof: (i) ⇒ we have $d_1(x,y) = \frac{d(x,y)}{1+d(x,y)}$.

(a) Since $d(x,y) \geq 0 \quad \forall x,y \in X$, then we have $d_1(x,y) \geq 0 \quad \forall x,y \in X$

(b) when $y = x$, then

$$d_1(x,x) = \frac{d(x,x)}{1+d(x,x)} = 0 \quad [\text{using (ii)}]$$

~~Next~~ $d_1(x,y) = 0$

ie, $\frac{d(x,y)}{1+d(x,y)} = 0$

⇒ $d(x,y) = 0$

⇒ $x = y$ [using (ii)]

∴ $d_1(x,y) = 0$ iff $x = y$.

(c) $$d_1(x,y) = \frac{d(x,y)}{1+d(x,y)}$$

$$= \frac{d(y,x)}{1+d(y,x)} \quad [\because d(x,y) = d(y,x) \ \forall x,y \in X.]$$

$$= d_1(y,x)$$

∴ $d_1(x,y) = d_1(y,x) \quad \forall x,y \in X$.

(d) We have from (iv), for $x, y, z \in X$,

$$d(x,y) \le d(x,z) + d(z,y)$$

$\therefore$ $1 + d(x,y) \le 1 + d(x,z) + d(z,y)$

or. $1 - \dfrac{1}{1+d(x,y)} \le 1 - \dfrac{1}{1+d(x,z)+d(z,y)}$

a. $\dfrac{d(x,y)}{1+d(x,y)} \le \dfrac{d(x,z)+d(z,y)}{1+d(x,z)+d(z,y)}$

a $\dfrac{d(x,y)}{1+d(x,y)} \le \dfrac{d(x,z)}{1+d(x,z)} + \dfrac{d(z,y)}{1+d(z,y)}$

or, $d_1(x,y) \le d_1(x,z) + d_1(z,y)$

$\therefore$ $d_1(x,y)$ satisfy all the conditions of the distance functions. then $d_1(x,y)$ is a metric.

(II) $\Rightarrow$ (a) we have $d_2(x,y) = \min\{1, d(x,y)\}$
since $d(x,y) \geq 0$ $\forall x,y \in X$, then
$d_2(x,y) \geq 0$ $\forall x,y \in X$

(b) As $d(x,y) = 0$ iff $x = y$.

then $d_2(x,x) = \min\{1, d(x,x)\}$
$= \min\{1, 0\}$
$= 0$.

$\therefore$ $d_2(x,y) = 0$, if $x = y$.

let $d_2(x,y) = 0$, then we have
$\min\{1, d(x,y)\} = 0$
$\Rightarrow d(x,y) = 0$
$\Rightarrow x = y$ [using (ii)]

$\therefore$ $d_2(x,y) = 0 \iff x = y$.

Scanned by CamScanner

(c) We have

$$d_2(x,y) = \min\{1, d(x,y)\}$$
$$= \min\{1, d(y,x)\} \quad , \text{[using (iii)]}$$
$$= d_2(y,x)$$

$\therefore \quad d_2(x,y) = d_2(y,x) \quad \forall\, x, y \in X.$

(d) Let $x, y, z \in X$, then

<u>Case-I</u> :- If $d(x,y) \geq 1$ or $d(y,z) \geq 1$, then we have

$$d_2(x,y) + d_2(y,z) \geq 1, \quad \text{while}$$
$$d_2(x,z) = \min\{1, d(x,z)\} \leq 1.$$

Thus in this case

$$d_2(x,z) \leq d_2(x,y) + d_2(y,z)$$

<u>Case-II</u> :- If $d(x,y) < 1$ and $d(y,z) < 1$, then

$$d(x,z) \leq d(x,y) + d(y,z)$$
$$= \min\{1, d(x,y)\} + \min\{1, d(y,z)\}$$
$$= d_2(x,y) + d_2(y,z)$$

But $d_2(x,z) \leq d(x,z)$

Hence $d_2(x,z) \leq d_2(x,y) + d_2(y,z)$

$\forall\, x, y, z \in X$

$d_2(x,y)$ satisfy all the conditions of the distance function then d_2 is a metric on X.

<u>Ex 19</u> (1) For $x, y \in K$ (R or C), define

$$d(x,y) = \min\{1, |x-y|\}$$

prove that d is a metric on K.

(2) Let $X = R$, for $x, y \in X$, define

(a) $d(x,y) = |x^2 - y^2|$ (b) $d(x,y) = |\sin(x-y)|$,

then prove that (X, d) is not a metric space.

Ex! 10 Let d and d^* be metrics defined on a Set X. prove that

(a) $d+d^*$ is a metric on X, In particular, $2d$ is a metric on X.

(b) $\max\{d, d^*\}$ is a metric on X.

Soln: Since d and d^* be metrics, then they Satisfy all the conditions of the distance ~~functions~~

i) $d(x,y) \geq 0$ and $d^*(x,y) \geq 0 \quad \forall x, y \in X$

ii) $d(x,y) = 0$ and $d^*(x,y) = 0$ iff $x = y$

iii) $d(x,y) = d(y,x)$
$d^*(x,y) = d^*(y,x) \quad \forall x, y \in X$

iv) $d(x,y) \leq d(x,z) + d(z,y)$
and $d^*(x,y) \leq d^*(x,z) + d^*(z,y)$
$\forall x, y, z \in X$

Proof(a) i) Let $\sigma_1 = d+d^*$.

Since $d(x,y) \geq 0$ and $d^*(x,y) \geq 0, \forall x, y \in X$

then we have
$$\sigma_1(x,y) = d(x,y) + d^*(x,y) \geq 0$$

$\therefore \sigma_1(x,y) \geq 0, \forall x, y \in X$

ii) Also, when $y = x$, then
$$\sigma_1(x,x) = d(x,x) + d^*(x,x) = 0$$

let $\sigma_1(x,y) = 0$
i.e, $d(x,y) + d^*(x,y) = 0$
$\Rightarrow d(x,y) = 0$ and $d^*(x,y) = 0$
$\Rightarrow x = y$ [using (ii)]

$\therefore \sigma_1(x,y) = 0$ iff $x = y$

(iii) We have

$$\sigma_1(x,y) = d(x,y) + d^*(x,y)$$
$$= d(y,x) + d^*(y,x) \quad [\text{using (iii)}]$$
$$= \sigma_1(y,x)$$

(iv) Let $x,y,z \in X$,

$$\sigma_1(x,y) = d(x,y) + d^*(x,y)$$
$$\le \{d(x,z) + d(z,y)\} + \{d^*(x,z) + d^*(z,y)\}$$
$$= \{d(x,z) + d^*(x,z)\} + \{d(z,y) + d^*(z,y)\}$$
$$= \sigma_1(x,z) + \sigma_1(z,y)$$

$\therefore \sigma_1(x,y) \le \sigma_1(x,z) + \sigma_1(z,y) \quad \forall x,y,z \in X.$

$\therefore \sigma_1(x,y)$ satisfy all the conditions of the distance function, then $\sigma_1(x,y)$ i.e. $(d + d^*)$ is a metric on X.

Soln (b) $\Rightarrow$ Let $\sigma_2(x,y) = \max\{d, d^*\}$.

(i) Since $d(x,y) \ge 0$ and $d^*(x,y) \ge 0 \quad \forall x,y \in X$

then $\sigma_2(x,y) = \max\{d, d^*\} \ge 0 \quad \forall x,y \in X$

(ii) Also we see that $\sigma_2(x,y) = 0$ iff $x = y$.

(iii) $$\sigma_2(x,y) = \max\{d(x,y), d^*(x,y)\}$$
$$= \max\{d(y,x), d^*(y,x)\}$$
$$= \sigma_2(y,x)$$

(iv) We have, for any $x,y,z \in X$.

$$d(x,y) \le d(x,z) + d(z,y)$$
$$\le \max\{d(x,z), d^*(x,z)\} + \max\{d(z,y), d^*(z,y)\}$$
$$= \sigma_2(x,z) + \sigma_2(z,y)$$

Similarly, $d^*(x,y) \le \sigma_2(x,z) + \sigma_2(z,y)$

Thus $$\sigma_2(x,y) = \max\{d(x,y), d^*(x,y)\}$$
$$\le \sigma_2(x,z) + \sigma_2(z,y) \quad \forall x,y,z \in X$$

$\therefore \sigma_2(x,y)$ satisfy all the conditions of the distance function then $\sigma_2(x,y)$ is a metric on X.

Scanned by CamScanner

Show that (X, d) is a metric space, where (7)
$X = N$ and $d : X \times X \to R$ is given by

$$d(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right|$$

Ex! 12 Let (X, d) be a metric space and let K be a fixed positive real numbers, for $x, y \in X$, define

$$d^*(x,y) = K \cdot d(x,y).$$

prove that d^* is a metric on X, and the ordered pair (X, d^*) is a metric space on X. [2H-10]

Ex! 13 For $x, y \in X$, define $d(x,y) = \begin{cases} 0 & , x = y \\ |x| + |y| & , x \neq y \end{cases}$,

Prove that d is a metric on X.

Th! Let (X, d) be a metric space, prove that

$$|d(x,z) - d(z,y)| \leq d(x,y), \quad \text{for all } x, y, z \in X.$$

[VH-11]

Proof! We apply triangle inequality to $d(x,z)$ and $d(z,y)$ separately,

We have
$$d(x,z) \leq d(x,y) + d(y,z) \quad \text{—— (1)}$$

and
$$d(z,y) \leq d(z,x) + d(x,y) \quad \text{—— (2)}$$

or, $d(z,y) \leq d(x,z) + d(x,y)$

Now, $d(x,z) - d(z,y) \geq -d(x,y)$

$\cancel{|d(x,z) - d(z,y)|}$

$\leq \cancel{|d(x,y) + d(y,z) - d(z,x) - d(x,y)|}$

$= \cancel{|d(y,z) - d(z,x)|}$

from (1), we have,

$$d(x,z) \leq d(x,y) + d(y,z)$$

$$\Rightarrow d(x,z) \leq d(x,y) + d(z,y) \qquad \text{[As } d \text{ is a metric on } X, \ d(z,y) = d(y,z)\text{]}$$

$$\Rightarrow d(x,z) - d(z,y) \leq d(x,y) \quad \text{—— (3)}$$

from (2), $\{d(x,z) - d(z,y)\} \geq \{-d(x,y)\}$ —— (4)

$\therefore$ From (3) and (4), $\Rightarrow |d(x,z) - d(z,y)| \leq d(x,y)$ [as $d(x,y) \geq 0$ $\forall x, y \in X$].

(proved)

Scanned by CamScanner

Ex: 14 Let X consist of all real-valued continuous functions $x(t)$ defined on the closed interval $[a,b]$. If $x(t), y(t) \in X$, we define

$$d(x(t), y(t)) = \sup_{a \le t \le b} |x(t) - y(t)|,$$

Show that (X,d) is a metric space. This space is denoted by $C[a,b]$, and this metric d on $C[a,b]$ is called the Uniform metric.

Soln:-

(I) We first note that for $x, y \in C[a,b]$, $x-y \in C[a,b]$ and since the supremum of any continuous functions on a closed interval exists, $d(x,y)$ is finite. clearly $d(x,y) \geq 0$ and (II) since suprimum of a non-negative function vanishes iff the function vanishes identically, it follows that $d(x,y)=0$ iff $x(t) = y(t)$, $\forall\ t \in [a,b]$. i.e, $x = y$.

(III) Also, for any $x, y \in C[a,b]$, obviously $d(x,y) = d(y,x)$.

(IV) For proving the triangle inequality, let $x, y, z \in C[a,b]$. Then for any $t \in [a,b]$

$$|x(t) - y(t)| \leq |x(t) - z(t)| + |z(t) - y(t)|$$

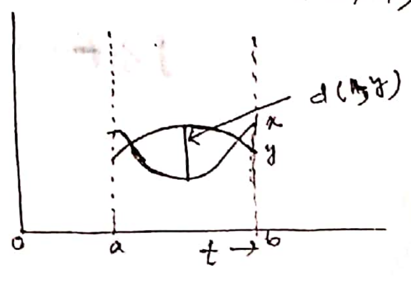
$$\leq \sup_{t \in [a,b]}(|x(t) - z(t)|) + \sup_{t \in [a,b]}(|z(t) - y(t)|)$$

$$= d(x,z) + d(z,y)$$

Then $\sup_{t \in [a,b]} (|x(t) - y(t)|) \leq d(x,z) + d(z,y)$.

Hence $d(x,y) \leq d(x,z) + d(z,y)$

This shows that (X,d) is a metric space.



Ex:(15) Let $B[a,b]$ be the set of all real valued functions defined and bounded on $[a,b]$. (8)
For $x, y \in B[a,b]$, define

$$d(x,y) = \sup_{t\in[a,b]} |x(t) - y(t)|$$

then prove that $(B[a,b], d)$ is a metric space.

Ex! 16 Prove that the set $C[a,b]$ of all real-valued functions continuous on the interval $[a,b]$ with the function d defined by

$$d(f,g) = \left(\int_a^b (f(x) - g(x))^2 dx\right)^{1/2}$$

is a metric space.

Sol^n:⇒ To establish triangle inequality we need the following:
We consider the function, for $t \in [a,b]$

$$\varphi(t) = \int_a^b (t\cdot f(x) + g(x))^2 . dx$$

$$= \int_a^b (t^2 f^2(x) + 2t f(x)\cdot g(x) + g^2(x))\, dx$$

$$= t^2 \int_a^b f^2(x)\, dx + 2t \int_a^b f(x) g(x)\, dx + \int_a^b g^2(x) dx .$$

Since $\varphi(t) \geq 0 \;\; \forall\, t \in [a,b]$, therefore the discriminate of the quadratic in t should be non-positive, and so

$$\left(\int_a^b f(x)\cdot g(x)\, dx\right)^2 \leq \int_a^b f^2(x)\, dx \cdot \int_a^b g^2(x)\, dx .$$

ie, $$\left(\int_a^b f(x)\cdot g(x)\, dx\right) \leq \left(\int_a^b f^2(x)\, dx\right)^{1/2} \cdot \left(\int_a^b g^2(x)\, dx\right)^{1/2} \quad \text{—— (1)}$$

Now we consider,

$$\left[\left(\int_a^b (f(x) - h(x))^2\right)^{1/2} + \left(\int_a^b (h(x) - g(x))^2\right)^{1/2}\right]^2$$

$$= \int_a^b (f(x) - h(x))^2 dx + \int_a^b (h(x) - g(x))^2 dx$$
$$+ 2\left(\int_a^b (f(x) - h(x))^2 dx\right)^{1/2} \cdot \left(\int_a^b (h(x) - g(x))^2 dx\right)^{1/2}$$

Scanned by CamScanner

$\geq \int_a^b (f(x) - h(x))^2 dx + \int_a^b (h(x) - g(x))^2 dx$

$+ 2\int_a^b (f(x) - h(x))(h(x) - g(x))\, dx$

$= \int_a^b (f(x) - h(x) + h(x) - g(x))^2 dx$

$= \int_a^b (f(x) - g(x))^2 dx$.

Hence

$$\left(\int_a^b (f(x) - g(x))^2 dx\right)^{1/2} \leq \left(\int_a^b (f(x) - h(x))^2 dx\right)^{1/2} + \left(\int_a^b (h(x) - g(x))^2 dx\right)^{1/2}$$

$\therefore \quad d(f,g) \leq d(f,h) + d(h,g)$

The other three conditions we can easily prove.

Hence $d(f,g)$ is a metric and $(C[a,b], d)$ is a metric space.

<u>Ex! 17</u> Show that the set R^n with d^* defined by $d^*(x,y) = \sum_{i=1}^{n} |x_i - y_i|$, is a metric space. ($d^*$ is called the rectangular metric on R^n). [V.H-11]

<u>Soln:</u> ⇒ (iv) For triangle inequality, we consider

$$d^*(x,y) = \sum_{i=1}^{n} |x_i - y_i| = \sum_{i=1}^{n} |x_i - z_i + z_i - y_i|$$

$$\leq \sum_{i=1}^{n} |x_i - z_i| + \sum_{i=1}^{n} |z_i - y_i|$$

$$= d^*(x,z) + d^*(z,y)$$

$\therefore \quad d^*(x,y) \leq d^*(x,z) + d^*(z,y) \;\; \forall\, x,y,z \in R^n$

(iii) We have

$$d^*(x,y) = \sum_{i=1}^{n} |x_i - y_i| = \sum_{i=1}^{n} |y_i - x_i|$$

$$= d^*(y,x)$$

$\therefore \quad d^*(x,y) = d^*(y,x) \quad \forall\, x,y \in R^n$.

(ii) When $x = y$, i.e., $x_i = y_i$, $\forall i = 1, 2, \ldots n$. (7)

then $d^*(x, x) = \sum_{i=1}^{n} |x_i - x_i|$

$= 0$.

Let $d^*(x, y) = 0 \Rightarrow \sum_{i=1}^{n} |x_i - y_i| = 0$

$\Rightarrow x_i - y_i = 0 \quad \forall i = 1, 2, \ldots n$

$\Rightarrow x_i = y_i, \forall i = 1, 2, 3, \ldots n$

$\Rightarrow x = y$.

$\therefore d^*(x, y) = 0$ iff $x = y$.

(i) Also we have

$d^*(x, y) = \sum_{i=1}^{n} |x_i - y_i| \geq 0 \quad \forall x, y \in R^n$.

$\therefore d^*(x, y)$ satisfy all the conditions of the distance function, then d^* is a metric on R^n. and (R^n, d^*) is a metric space.

Ex: 18 Let X be a non-empty set and a ~~function~~ d from $X \times X \to R$ into R satisfy :-

(i) $d(x, y) = 0$ if and only if $x = y$

and (ii) $d(x, y) \leq d(x, z) + d(y, z)$, $\forall x, y, z \in X$. [VH-09]

Prove that (X, d) is a metric space.

Proof/ We take $y = x$ in (ii), then we have

$d(x, x) \leq d(x, z) + d(x, z)$

$\Rightarrow 2d(x, z) \geq 0$, [using (i)]

$\Rightarrow d(x, z) \geq 0$

again putting $x = z$ in (ii), we have.

$d(z, y) \leq d(z, z) + d(y, z)$

$\Rightarrow d(z, y) \leq d(y, z)$ —— (1)

If we interchange the role of y and z in (1)

then we get $d(y, z) \leq d(z, y)$ —— (2)

from ① and ②, it follows that

$$d(y,z) = d(z,y) \quad \forall\ z,y \in X$$

$\therefore$ (X,d) is a metric space.

EX: For $x,y \in K$ define $d(x,y) = \begin{cases} 0, & x=y \\ |x|+|y|, & x \neq y \end{cases}$

Prove that d is a metric on K.

Sol^n i) Since $|x|+|y| \geq 0 \ \forall x,y \in K$

then $d(x,y) \geq 0 \ \forall x,y \in K$

ii) let $x=y$

$\therefore d(x,y) = 0$ [by the given function]

Conversely

let $d(x,y) = 0$

$\Rightarrow x=y$ [by the given function]

iii) $d(x,y) = \begin{cases} 0 & x=y \\ |x|+|y|, & x\neq y \end{cases}$

$= \begin{cases} 0 \\ |y|+|x|, & y \neq x \end{cases}$

$= d(y,x)$

$\therefore d(x,y) = d(y,x) \ \forall x,y \in K$

iv) let $x,y,z \in K$

	$d(x,y)$	$d(y,z)$	$d(x,z)$	$d(x,y)+d(y,z)$	
$x=y=z$	0	0	0	0	$=$
$x\neq y=z$	$\lvert x\rvert+\lvert y\rvert$	0	$\lvert x\rvert+\lvert z\rvert$	$\lvert x\rvert+\lvert y\rvert$	$=$
$x=y\neq z$	0	$\lvert y\rvert+\lvert z\rvert$	$\lvert x\rvert+\lvert z\rvert$	$\lvert y\rvert+\lvert z\rvert$	$=$
$z\neq x\neq y$	$\lvert x\rvert+\lvert y\rvert$	$\lvert y\rvert+\lvert z\rvert$	$\lvert x\rvert+\lvert z\rvert$	$\lvert x\rvert+2\lvert y\rvert+\lvert z\rvert$	$\leq$
$z=x\neq y$	$\lvert x\rvert+\lvert y\rvert$	$\lvert z\rvert+\lvert y\rvert$	0	$\lvert x\rvert+2\lvert y\rvert+\lvert z\rvert$	$\leq$

from the above table we have

$d(x,z) \leq d(x,y) + d(y,z) \ \forall x,y,z \in K$

Since d satisfy all the condition of distance function, then ~~the~~ d is a metric on K